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# Some exact statistics of two-dimensional viscous flow with random forcing

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By regarding the amplitudes of a set of orthogonal modes as the co-ordinates in an infinite-dimensional phase space, the probability distribution for an ensemble of randomly forced two-dimensional viscous flows is determined as the solution of the continuity equation for the phase flow. For a special, but infinite, class of types of random forcing, the exact equilibrium probability distribution can be found analytically from the Navier–Stokes equations. In these cases, the probability distribution is the product of exponential functions of the integral invariants of unforced inviscid flow.

## 1. Introduction

During the past few years, increasing attention and effort have been directed towards the study of two-dimensional turbulent flows, not because they are realizable in the laboratory or strictly observable physically, but because they are simple and plausible models of the quasi-horizontal turbulent flow of the stably stratified atmospheres of the earth and other fairly rapidly rotating planets.

It has long been recognized that there is a fundamental distinction between two- and three-dimensional turbulence. In three-dimensional motion, the total kinetic energy is the only known integral invariant for unforced inviscid flow.† In two-dimensional inviscid flow, however, there are two basic invariants, namely, the total kinetic energy and variance of vorticity (enstrophy). As Fjörtoft (1953) pointed out, the latter fact implies that the kinetic energy of two-dimensional motion must be transferred simultaneously from both large to small and small to large scales by nonlinear interactions between different scales of motion, whereas there is a unidirectional nonlinear cascade of energy from large to small scales of motion in three-dimensional flow.

This peculiarity of two-dimensional flow has lead Batchelor (1969), Kraichnan (1967) and Leith (1968) to postulate the existence of two distinct 'inertial' subranges of turbulence, above and below some energy-receiving part of the spectrum. The small-scale subrange is characterized by negligible turbulent transfer of energy but strong enstrophy transfer to small scales; the large-scale subrange

† Excepting the so-called 'helicity' integral, which vanishes identically in isotropic turbulence.

is one in which turbulent enstrophy transfer is negligible, but the nonlinear transfer of energy is directed from smaller to larger scales of motion.

The inferred kinetic energy spectra in the two subranges are also distinctly different. According to dimensional arguments analogous to those of Kolmogoroff (1941) and Batchelor (1946) for homogeneous isotropic turbulence in three dimensions, the energy density in the large-scale inertial subrange (above the energy-receiving scales) might be expected to vary as the minus five-thirds power of scalar wavenumber, whereas it should vary as the minus three power of wavenumber in the small-scale inertial subrange. These conjectures have been confirmed to a remarkable degree by the results of numerical experiments carried out by Lilly (1969), particularly for the small-scale inertial subrange, and by the more recent numerical calculations of Fox & Orszag (1972).

It has become apparent, however, that the results of approximate numerical integrations of the Navier–Stokes equations (whether by Eulerian finite-difference methods or by truncated orthogonal representations) leave some doubt as to the validity of conclusions drawn from numerically computed statistics of the small-scale features of turbulent flows. To a considerable extent, this uncertainty arises from the lack of any exact standard against which one can judge the accuracy of approximate numerical calculations that may be almost equally inexact over a large range of mesh sizes or numbers of representative orthogonal modes.

The purpose of this paper, briefly, is to show that some exact statistics of randomly forced, two-dimensional viscous flows can be derived analytically from the Navier–Stokes equations. Although these solutions certainly do not comprise a general theory of two-dimensional turbulence, they do provide a statistical standard with which approximate numerical results may be compared.

## 2. The physical and mathematical problem

We shall be concerned with the two-dimensional flow of a homogeneous and incompressible, but viscous fluid whose motion is forced by a random distribution of sources and sinks of vorticity. In this case, the Navier–Stokes equations reduce to the vorticity equation:

$$\frac{\partial}{\partial t}\nabla^2\psi + \mathbf{n}.\nabla\psi \times \nabla\nabla^2\psi - \nu\nabla^2\nabla^2\psi = F(x,y,t), \tag{1}$$

in which $x$ and $y$ are Cartesian co-ordinates in the plane of motion, $\psi$ is the stream function, $\nabla$ and $\nabla^2$ are the vector gradient and Laplacian operators in the plane of motion, $\mathbf{n}$ is the unit vector normal to that plane, $\nu$ is the coefficient of kinematic viscosity and $F(x,y,t)$ is a randomly varying function whose properties will be specified later.

From the standpoint of treating the statistics of an ensemble of solutions of (1), it is convenient to represent the stream function $\psi$ as a linear combination of orthogonal functions, i.e. as

$$\psi(x,y,t) = \sum_{i=1}^{\infty} A_i(t)\,\phi_i(x,y), \tag{2}$$

where the eigenfunctions $\phi_i$ are solutions of

$$\nabla^2\phi_i = -\alpha_i^2\phi_i \tag{3}$$

subject to the condition that the $\phi_i$ vanish on the boundaries of a finite but large rectangular subdomain of a flow that is periodic in both $x$ and $y$ in the infinite domain. Equation (3), together with Gauss' theorem, implies that the $\phi_i$ are orthogonal, since they vanish on the boundaries. The $\alpha_i$ are clearly discrete eigenvalues. Thus, we can avoid summation over many indices by requiring that there be a unique $\phi_i$ corresponding to each $\alpha_i$. This condition is fulfilled, for example, if the length $L$ and width $W$ of the rectangular subdomain is such that $L^2/W^2$ is an irrational number, even if that ratio is virtually indistinguishable from unity. Finally, since (3) is homogeneous, we may normalize the $\phi_i$ so that

$$\int_A \phi_i^2\, dA = A, \tag{4}$$

where $A$ is the area of the subdomain and the integral is taken over that area.

The evolution equations for the amplitude factors $A_i(t)$ are derived by substituting the general representation given by (2) into the vorticity equation (1), noting that the product of summations over a single index may be written as a double summation of products over two indices:

$$\sum_{i=1}^{\infty} \alpha_i^2 \frac{dA_i}{dt}\phi_i + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_j^2 A_i A_j \mathbf{n}.\nabla\phi_i \times \nabla\phi_j + \nu \sum_{i=1}^{\infty} \alpha_i^4 A_i \phi_i + F(x,y,t) = 0.$$

Thus, on multiplying the equation above by a particular eigenfunction $\phi_k$, integrating over the entire subdomain $A$ and introducing the conditions that the $\phi_i$ are orthogonal and normalized, we have

$$\alpha_k^2 \frac{dA_k}{dt} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \beta_{ijk}\alpha_j^2 A_i A_j + \nu\alpha_k^4 A_k = \alpha_k f_k(t), \tag{5}$$

where

$$\beta_{ijk} = -\frac{1}{A}\int_A \phi_k \mathbf{n}.\nabla\phi_i \times \nabla\phi_j\, dA \tag{6}$$

and

$$\alpha_k f_k(t) = -\frac{1}{A}\int_A \phi_k F(x,y,t)\, dA.$$

Several distinctive and important properties of the nonlinear interaction coefficients $\beta_{ijk}$ follow directly from (6) and the boundary conditions on the $\phi_i$. By integrating by parts, one can readily verify that $\beta_{ijk}$ vanishes if any two indices are equal, is invariant under cyclic permutation of indices and reverses sign under non-cyclic permutation of indices.

To display the symmetries and asymmetries of (5) more clearly, it is useful to consider a new set of dependent variables, namely, $X_k = \alpha_k A_k$. Under this transformation, (5) becomes

$$\dot{X}_k = N_k - \nu\alpha_k^2 X_k + f_k(t), \tag{7}$$

in which

$$N_k = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\alpha_j^2 \beta_{ijk}}{\alpha_i \alpha_j \alpha_k} X_i X_j \tag{8}$$

and $\dot{X}_k \equiv dX_k/dt$. It will be noted that $X_k^2$ is just the kinetic energy per unit mass contained in the $k$th mode.

## 3. Statistical formulation of the problem

Introducing the simplest ideas of statistical mechanics, we now regard the $X_k$ as the co-ordinates of a point in an infinite-dimensional phase space, corresponding to the state of one particular 'realization' of the system at one moment in time. The time evolution of a single realization is then described by the trajectory of a single imaginary point, or 'particle', travelling through phase space in accordance with the parametric equations of motion given by (7).

Let us next suppose that a very large ensemble of realizations, large enough for the density of 'particles' in the phase space to be quasi-continuous, were initiated at the same time. In general, of course, the density of such 'particles' varies with time and position in the phase space. No new realizations, however, are created after the initial time, so that the number of 'particles' within a volume element $\prod_{k=1}^{\infty} dX_k$ must increase or decrease at a rate equal to the rate of net transport of particles into or out of the volume element. Thus, since $\dot{X}_k$ is the component of velocity in the $X_k$ direction in the phase space, the continuity equation for conservation of 'particles' is

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{\infty} \frac{\partial}{\partial X_k} (\rho \dot{X}_k) = 0, \tag{9}$$

where $\rho$ is the density of 'particles' or realizations in the phase space. The function $\rho$ is usually called the probability distribution, simply because the probability that any one of an ensemble of realizations (or 'particles') lies within a volume element $\prod_{k=1}^{\infty} dX_k$ in the neighbourhood of $X_k$ is just

$$\rho(X_k, t) \prod_{k=1}^{\infty} dX_k.$$

Thus, if $\rho(X_k, t)$ is known, this relation enables us to calculate the ensemble average of any function of the $X_k$. In particular,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \rho(X_k, t) \prod_{k=1}^{\infty} dX_k = 1.$$

Substituting for $\dot{X}_k$ from (7) into (9), we see that

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{\infty} \frac{\partial}{\partial X_k} (\rho N_k) - \nu \sum_{k=1}^{\infty} \frac{\partial}{\partial X_k} (\rho \alpha_k^2 X_k) + \sum_{k=1}^{\infty} \frac{\partial}{\partial X_k} (\rho f_k) = 0. \tag{10}$$

It is worth noting that this equation has only one dependent variable, namely, the probability distribution $\rho$, and that it is linear.

Let us next consider the form of $\rho f_k$, which is to be interpreted as the net rate of transport of 'particles' in the $X_k$ direction, due solely to the displacements of a very large ensemble of 'particles' moving with randomly varying speeds $f_k$ in the $X_k$ direction. It is intuitively evident (and is easily shown) that the transport of 'particles' by random motions is a diffusive process, such that in the ensemble average

$$\rho f_k = -\mu_k \partial \rho / \partial X_k,$$

where the coefficient of diffusion $\mu_k$ is proportional to

$$\langle f_k^2 \rangle \int_0^{t_0} \Phi_k(\tau)\, d\tau.$$

The pointed brackets denote the ensemble average, and $\Phi_k(\tau)$ is the normalized autocorrelation function for $f_k(t)$. The constant $t_0$ is large compared with the time scale of random fluctuations of $f_k(t)$, but is small compared with the time scale of fluctuations of $\rho$. Thus $t_0$ may be considered as infinite.

An additional simplification of (10) may be made by noting that

$$\partial N_k / \partial X_k = 0,$$

simply because $\beta_{ikk} = \beta_{kjk} = 0$. Thus (10) reduces to

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{\infty} N_k \frac{\partial \rho}{\partial X_k} - \nu \sum_{k=1}^{\infty} \frac{\partial}{\partial X_k} (\rho \alpha_k^2 X_k) - \sum_{k=1}^{\infty} \mu_k \frac{\partial^2 \rho}{\partial X_k^2} = 0. \tag{11}$$

To make (11) more symmetrical in its independent variables, we next carry out a simple linear transformation:

$$X_k = \mu_k^{\frac{1}{2}} Z_k,$$

whence (11) takes the form

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{\infty} M_k \frac{\partial \rho}{\partial Z_k} - \nu \sum_{k=1}^{\infty} \frac{\partial}{\partial Z_k} (\rho \alpha_k^2 Z_k) - \sum_{k=1}^{\infty} \frac{\partial^2 \rho}{\partial Z_k^2} = 0, \tag{12}$$

where, in view of (8),

$$M_k = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\alpha_j^2 \beta_{ijk}}{\alpha_i \alpha_j \alpha_k} \frac{(\mu_i \mu_j \mu_k)^{\frac{1}{2}}}{\mu_k} Z_i Z_j.$$

Thus far, we have suffered no loss of generality.

## 4. Some special equilibrium solutions

It will next be shown that, for certain special types of random forcing, (12) has equilibrium solutions of the form

$$\rho_0 = C \exp\left( -\tfrac{1}{2}\nu \sum_{k=1}^{\infty} \alpha_k^2 Z_k^2 \right), \tag{13}$$

in which $C$ is a constant, determined by the condition that the integral of $\rho_0$ taken over the entire phase space is unity. We first note that

$$\partial \rho_0 / \partial Z_k = -\nu \rho_0 \alpha_k^2 Z_k.$$

Thus, the third and fourth terms of (12) exactly cancel. Moreover, since $\rho_0$ is independent of $t$, the first term of (12) vanishes. Accordingly, the condition that $\rho_0$ be a solution of (12) is

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\alpha_j^2 \alpha_k^2 \beta_{ijk}}{\alpha_i \alpha_j \alpha_k} \frac{(\mu_i \mu_j \mu_k)^{\frac{1}{2}}}{\mu_k} Z_i Z_j Z_k = 0.$$

This condition is satisfied if there are constants $a$ and $b$ such that

$$\alpha_k^2 / \mu_k = a + b\alpha_k^2 \tag{14}$$

regardless of the independent variables $Z_k$, simply because the triple summation exhibited above vanishes identically, in virtue of the fact that the nonlinear interaction coefficients $\beta_{ijk}$ reverse sign under non-cyclic permutation of indices. To show this, we note that the triple summation may be reordered as

$$a \sum_{j=1}^{\infty} \alpha_j \mu_j^{\frac{1}{2}} Z_j \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{\beta_{ijk}}{\alpha_i \alpha_k} (\mu_i \mu_k)^{\frac{1}{2}} Z_i Z_k + b \sum_{i=1}^{\infty} \frac{\mu_i^{\frac{1}{2}} Z_i}{\alpha_i} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ijk} \alpha_j \alpha_k (\mu_j \mu_k)^{\frac{1}{2}} Z_j Z_k.$$

The first of these summations vanishes because the summand reverses sign with exchange of $i$ and $k$; the second vanishes since the summand reverses sign with exchange of $j$ and $k$.

With the restrictions stated in (14), the function given by (13) is an exact equilibrium solution of (12). It is not difficult to show that this solution is also stable and unique, by making use of the properties of (12), the condition of integrability and the fact that a diffusively controlled probability distribution cannot have singularities.

## 5. The partition of energy among randomly forced modes

The equilibrium spectrum of kinetic energy for a two-dimensional viscous flow, in which the statistical properties of random sources of vorticity are prescribed by (14), is easily calculated from (13). With the restriction (14), we may write the equilibrium probability distribution as †

$$\rho_0 = C \exp\left[-\tfrac{1}{2}\nu \sum_{k=1}^{\infty} (a + b\alpha_k^2) X_k^2\right]. \tag{15}$$

Thus, by definition, the ensemble average of the kinetic energy in the $p$th mode is

$$\langle X_p^2 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} X_p^2 \rho_0(X_k) \prod_{k=1}^{\infty} dX_k$$

$$= C \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} X_p^2 \prod_{k=1}^{\infty} \exp\left[-\tfrac{1}{2}\nu(a + b\alpha_k^2) X_k^2\right] dX_k$$

or, since

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \rho_0(X_k) \prod_{k=1}^{\infty} dX_k = 1,$$

$$\langle X_p^2 \rangle = \frac{\int_{-\infty}^{\infty} X_p^2 \exp\left[-\tfrac{1}{2}\nu(a + b\alpha_p^2) X_p^2\right] dX_p}{\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}\nu(a + b\alpha_p^2) X_p^2\right] dX_p}$$

$$= \frac{1}{\nu(a + b\alpha_p^2)} = \frac{\mu_p}{\nu \alpha_p^2}.$$

It should be pointed out that $b$ must be positive; $a$ may be negative, however, provided that $|a|^{\frac{1}{2}}/b^{\frac{1}{2}}$ is less than the smallest eigenvalue.

† Written in this form, the equilibrium probability distribution clearly depends only on the additive integral invariants of unforced inviscid two-dimensional flow, namely, total energy and enstrophy.

## 6. Conclusions and comment

The reader will undoubtedly have perceived that the solutions given by (15) are highly special and that, from the standpoint of a general theory of turbulence, they are not very interesting. This arises from the fact that the specification of random forcing is such that there is no *net* nonlinear transfer of energy into or out of any particular mode, in the ensemble average. That is not to say, however, that there is no nonlinear transfer between modes in any particular realization. For this reason, and because (15) yields exact statistical results, the solutions given here may prove a useful standard in judging the accuracy of results based on an ensemble of approximate numerical solutions.

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